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Meteoric Flux and Density Fields about a Finite Attractive Center Generated by a Stream Monoenergetic and Monodirectional at Infinity

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Abstract. Detailed meteoric flux and particle density field patterns generated by a meteoric stream, monodirectional and monoenergetic at infinity, incident upon a finite attractive center are derived and exhibited. Flux and density contours are obtained for a series of incident speeds spanning the entire range of established meteoric stream velocities. The development reveals the existence of five sectors about the attractive center. In some, flux is unscattered; in others, the flux is scattered; in one, the flux is both unscattered and scattered; in another, the *null cone,* there is no flux. All sectors including the null cone are treated in detail. An explicit theoretical explanation of abrupt discontinuities in flux and enhanced particulate concentrations near the earth, both of which are observed in rocket and satellite experiments, is given.

Author

INTRODUCTION

To establish a basis for the ensuing discussion, we need the following concepts (defined and discussed in greater detail by *Shelton et al.* [1964] and in the accompanying paper by *Hale and Wright* [1964]): that about any attractive center an incident monoenergetic, monodirectional stream (MED) will define a surface whose points constitute the locus of perigee for the entire family of trajectories resulting from the physical configuration. Clearly, this surface is a boundary between a zone of flux approaching the attractive center on one side and a zone of flux receding from the attractive center on the other side. In our development this surface is called the $\theta_{k\,max}$ surface; we shall assume that the reader has access to the papers cited above.

Another concept, arising in the case of the finite attractive center, is that of the surface generated by the limiting trajectory of a given MED. That is, owing to the finite size of the attractive center, trajectories which would be bent around the center more than a certain limiting trajectory are intercepted by the center; the limiting trajectory is that trajectory which would be observed at infinity (i.e., as flux receding from the center) to have experienced the greatest deflection. Trajectories which would be bent more than the limiting trajectory [*Shelton et al.*, 1964, equations 43–47] never return to infinity by virtue of their colliding with and being absorbed by the attractive center. This limiting trajectory, when rotated about the direction of the initial stream velocity vector at infinity, defines a surface known as the limiting trajectory or θ_{lim} surface. Generally, the θ_{lim} surface has a node on the downstream axis of the pattern. Abrupt discontinuities in flux and density will be observed upon crossing this surface.

The θ_{lim} surface may be a boundary between scattered flux (ϕ_s) in the same hemisphere as it originated and a sector of no flux (the null 'cone' to be discussed subsequently). The θ_{lim} surface may also be a boundary between a sector containing ϕ_s flux only and a sector containing both ϕ_s and ϕ_s' scattered flux originating in the other of the two hemispheres defined by the plane containing V_∞ and perpendicular to the plane of a given trajectory.

As just mentioned, the field pattern formed by an MED incident upon a finite center gives rise to a sector of no flux, a region of complete shielding from the radiation. This sector, to be known as sector 4, is called the *null cone.* It forms on the back, or downstream, side of the attractive center, and its generatrix is the hyperbola of the limiting trajectory. Its apex is the node of the θ_{lim} surface. The surface of the null cone is swept through by the generatrix rotating through the node and along a circle on the surface of the spherical finite center; this

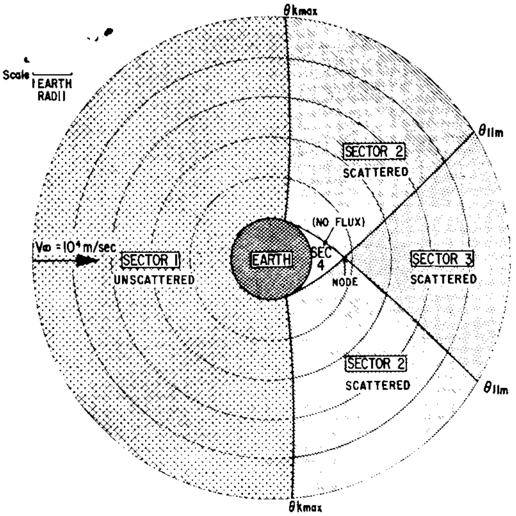

Fig. 1. Plot of sectors with θ_{1lm} and $\theta_{k\,max}$ surfaces for $V_\infty = 10^4$ m/sec. Particle stream assumed to be monodirectional and monoenergetic at infinity.

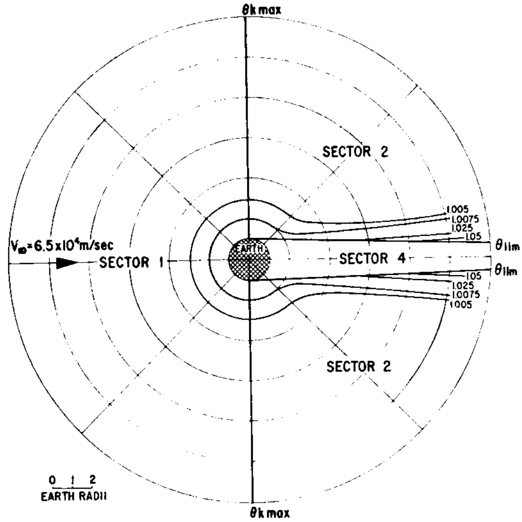

Fig. 2. Total particle flux contours relative to unit monodirectional, monoenergetic incident flux at infinity for $V_\infty = 65 \times 10^3$ m/sec about a finite attractive center of earth's mass.

circle is defined by the intersection of the $\theta_{k\,max}$ surface with the surface of the finite center.

For the reader's convenience in understanding this paper, we present a very brief discussion of the theory involved in forming the basic equation. The derivation of this equation and a great deal of peripheral material is treated in detail by *Shelton et al.* [1964].

If the value of the current at infinity between impact parameters a and $a + da$ is given by $J(\infty)$, then the value for the current between r and $r + dr$ is given by

$$-J(\bar{r})\cdot d\bar{A} = J(\bar{r})\ dA\ \cos\alpha = 2\pi a\ da J(\infty)$$

Writing the element of area dA in spherical coordinates as

$$dA = 2\pi r^2 \sin\theta\ d\theta$$

we have, after rearrangement,

$$\frac{J(r)}{J(\infty)} = \frac{a\ da}{r^2 \sin\theta\ \cos\alpha\ d\theta}$$

which is the basic equation used here for the development of the flux and density fields.

The impact parameter a is obtained from the well-known orbit equation [*Goldstein,* 1959]

$$r = \frac{ya^2}{1 + \sqrt{1 + y^2a^2}\ \cos(\theta - \theta_0)}$$

where

$$y = V_\infty^2/\gamma M$$

γ = gravitational constant.
M = mass of center of force.

The function $\cos\alpha$ is obtained directly from the conservation of angular momentum equation

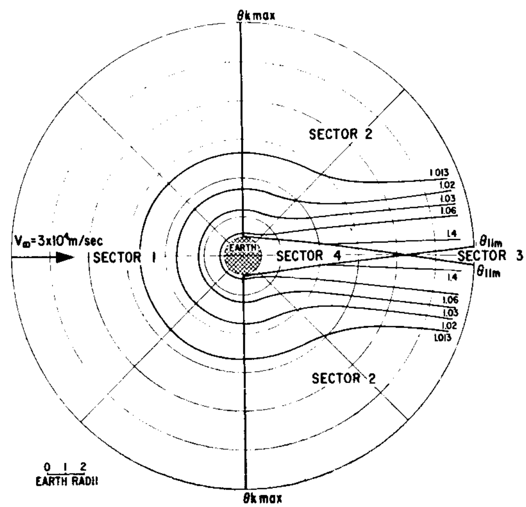

Fig. 3. Total particle flux contours relative to unit monodirectional, monoenergetic incident flux at infinity for $V_\infty = 3 \times 10^4$ m/sec about a finite attractive center of earth's mass.

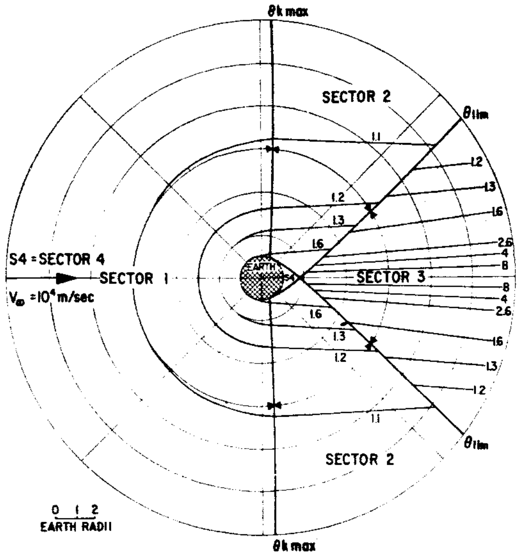

Fig. 4. Total particle flux contours relative to unit monodirectional, monoenergetic incident flux at infinity for $V_\infty = 10^4$ m/sec about a finite attractive center.

$$r\,V(r) \sin \alpha = V_\infty a$$

where α is the angle measured from the inward pointing radius vector to the directed trajectory defined to be positive in the counter clockwise sense.

Definition of Sectors

Study of Figure 1 should, with the following elaboration, make all the above definitions clear. Figure 1 is the sector pattern for $V_\infty = 10$ km/sec about the earth, and, though it does not contain all the features of such field patterns, it is thought to be the most typical case and is easily understood.

We see that:

Sector 1 (on the upstream or left-hand side of the figure) consists entirely of unscattered flux ϕ_D and is bounded by the $\theta_{k\,\max}$ surface.

Sector 2 is bounded by the $\theta_{k\,\max}$ surface on the upstream side and the $\theta_{1\mathrm{im}}$ surface on the downstream side and consists only of scattered flux ϕ_s originating within the same hemisphere in which it is observed.

Sector 3 is bounded entirely by the $\theta_{1\mathrm{im}}$ surface and is the region to the right, or on the downstream side, of the node in the $\theta_{1\mathrm{im}}$ surface. All flux in sector 3 is scattered and is formally noted by $\phi_s + \phi_s'$. The highest flux and particle densities occur in sector 3 along the axis (i.e., $\theta = \pi$, the direction that would be assumed by an unperturbed MED). On the axis, as in the case of the infinitesimal center [*Hale and Wright*, 1964], these concentrations become infinite, but in such a gentle manner that the flux or particle density within any finite volume is finite.

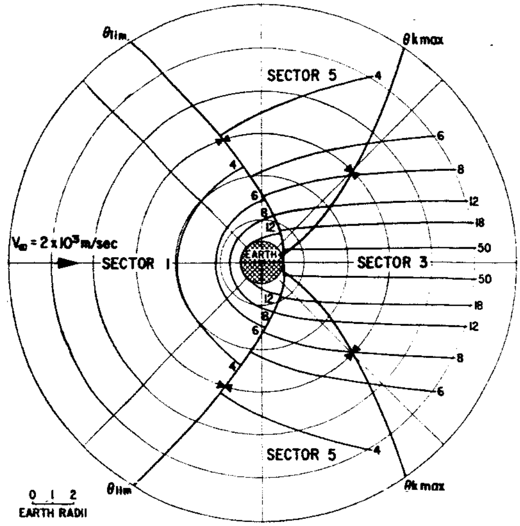

Fig. 5. Total particle flux contours relative to unit monoenergetic, monodirectional incident flux at infinity for $V_\infty = 2 \times 10^3$ m/sec about a finite attractive center.

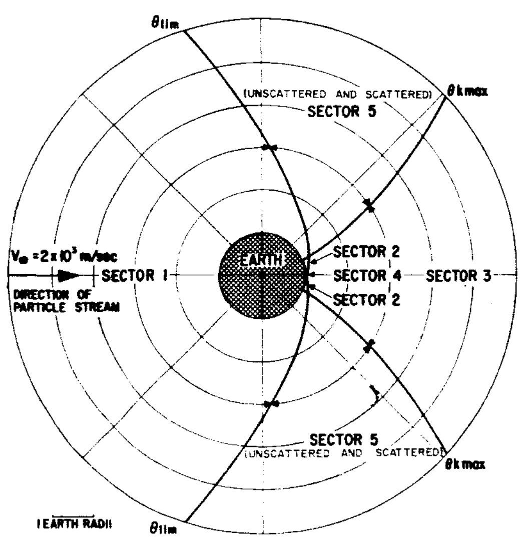

Fig. 6. Plot of sectors showing $\theta_{1\mathrm{im}}$ and $\theta_{k\,\max}$ surfaces for $V_\infty = 2 \times 10^3$ m/sec; particle stream assumed to be monodirectional and monoenergetic at infinity.

Because of our unrealistic assumption of an infinitely broad MED having persisted eternally, the flux or particle density within a finite volume on the axis would increase without limit as one moved farther downstream.

Sector 4 is a region bounded by the θ_{lim} surface and the surface of the finite center. It is the space enclosed on the upstream side of the θ_{lim} node. There is no radiation within sector 4 although, along its bounding surface in the vicinity of the node, flux and particle densities approach infinite values.

Sector 5, shown in Figure 6, occurs for lower energies and/or 'stronger' centers than those of Figure 1. If we examine, in order, the flux patterns of Figures 2, 3, 4, and 5 (i.e., proceeding from higher to lower energies) and notice in particular the flaring or bending toward upstream of the θ_{lim} surface, we will see that, in addition to an intersection of the θ_{lim} surface with the $\theta_{k\,\text{max}}$ surface along a circle on the finite center, there occurs, for energies below or center strength above a certain value (obtained subsequently), a second intersection of these two surfaces. This second intersection has not occurred in Figure 2; however, in Figure 5 the condition is so far advanced as to have enclosed and squeezed sector 2 down into an almost unrecognizable, extremely small generalized toroid centered about the θ_{lim} node. Furthermore, the new sector (sector 5), also a generalized toroid and centered along the circle defined by the second intersection, has appeared. Sector 5 extends to infinity and contains both unscattered flux ϕ_D and scattered flux ϕ_s'; the unscattered flux is proceeding downstream while the scattered flux is directed upstream. Figure 6, an expansion of Figure 5 with contours absent, should make the preceding discussion quite clear.

Derivation of $\theta_{k\,\text{max}}$ and θ_{lim} Surface Generatrices

The $\theta_{k\,\text{max}}$ surface is derived in the accompanying paper. Its equation is

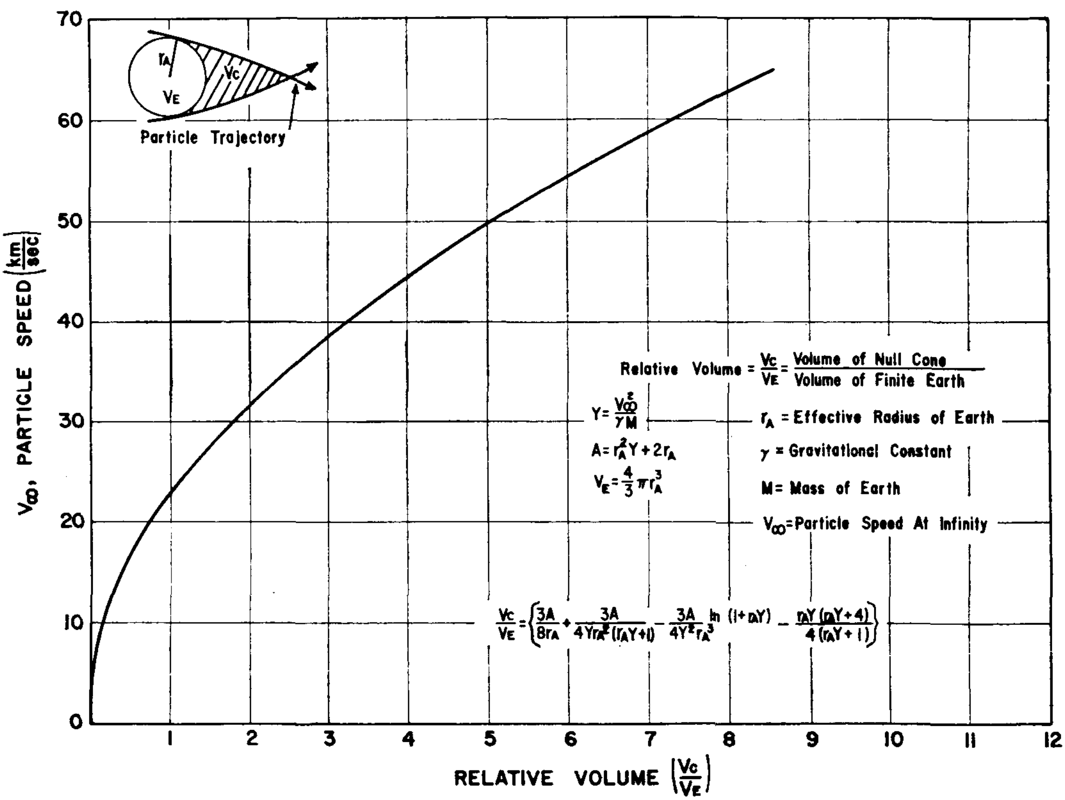

Fig. 7. Plot of the relative volume (V_c/V_E) of the null cone as a function of particle speed at infinity (V_∞).

$$\theta_{k\,max} = \text{arc cos}\left(\frac{-1}{\sqrt{1 + y^2a^2}}\right)$$

$$= \text{arc cos}\left(\frac{-1}{1 + yr}\right) \qquad (1)$$

where

$$y \equiv \frac{V_\infty^{\ 2}}{\gamma M} \qquad a^2 = r^2\left(1 + \frac{2}{ry}\right)$$

a = the impact parameter for an orbit whose perigee is r.
V_∞ = the particle speed at infinity.
r = the perigee distance from the center.
M = the mass of the attractive center.
γ = the gravitational constant.

The generatrix for the $\theta_{\lim}$ surface or the limiting trajectory is obtained directly [*Shelton et al.*, 1964, equation 66] from the orbit equation

$$r = \frac{ya^2}{1 + \sqrt{1 + y^2a^2}\cos(\theta - \theta_k)} \qquad (2)$$

where all quantities are as just defined, with the exception that r, θ now vary along the trajectory characterized by impact parameter a. If θ is to be $\theta_{\lim}$, that is, the orbit equation is to become the polar equation for the limiting trajectory, then the impact parameter a must be a_{max}, the maximum value of the impact parameter such that the associated trajectory is intercepted by, or grazes, the finite attractive center. For the earth the distance from the center to the point of closest approach is r_A (taken to be 6.53×10^6 meters including 1.28×10^5 meters of atmosphere). Thus, on the surface $\theta_{k\,max}$ is given by

$$\theta_{k\,max} = \text{arc cos}\left(\frac{-1}{\sqrt{1 + y^2a^2_{\ max}}}\right)$$

$$= \text{arc cos}\,\frac{-1}{1 + yr_A} \qquad (3)$$

where

$$a_{max}^{\ 2} = r_A^{\ 2}\left(1 + \frac{2}{yr_A}\right) \qquad (4)$$

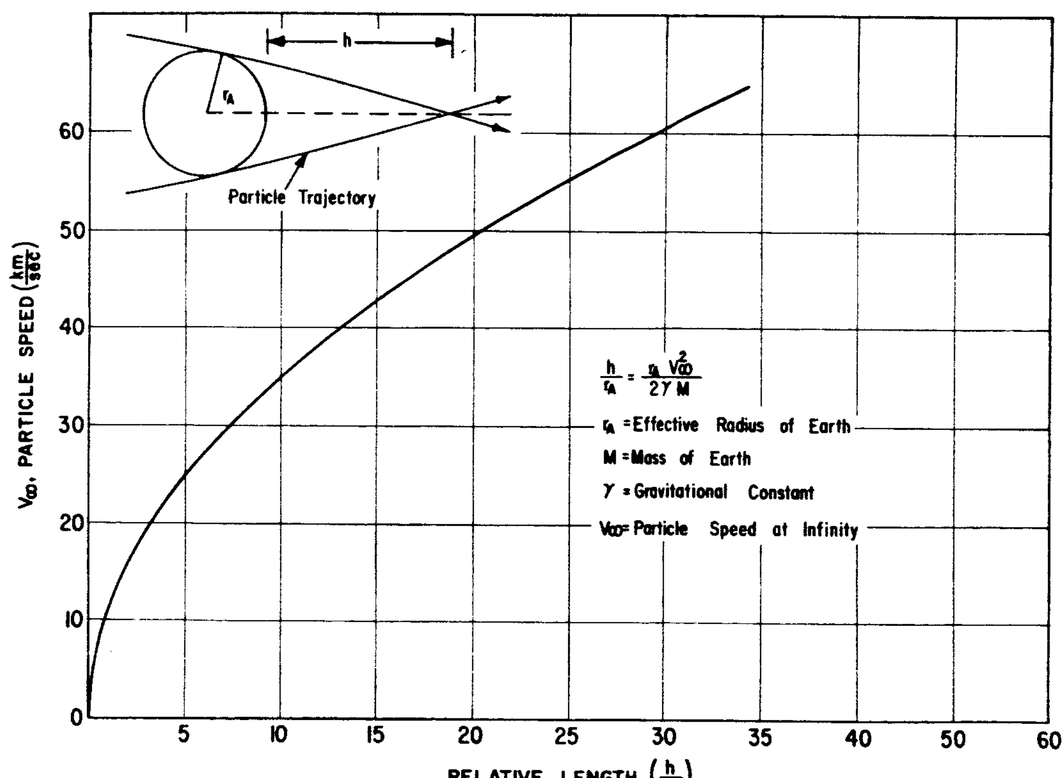

Fig. 8. Plot of the relative height (h/r_A) of the null cone as a function of particle speed at infinity (V_∞).

Making these substitutions and solving for $\theta \equiv \theta_{\lim}$ here, from the orbit equation 2,

$$\theta_{\lim} = \arc \cos\left(\frac{ya^2_{\max} - r}{r\sqrt{1 + y^2a^2_{\max}}}\right) + \arc \cos\left(\frac{-1}{\sqrt{1 + y^2a^2_{\max}}}\right) \quad (5)$$

which is simply the polar equation of the limiting trajectory.

The second intersection of the $\theta_{k\max}$ and $\theta_{\lim}$ surfaces occurs for a given y and $a_{\max}$ when $\theta_{k\max} = \theta_{\lim}$. The preceding development suggests we seek that value of $r_A\, y$ such that the second intersection occurs at infinity. Explicitly we demand

$$\lim_{r\to\infty} \theta_{\lim} = \lim_{r\to\infty} \theta_{k\max}$$

Now, from (1),

$$\lim_{r\to\infty} \theta_{k\max} = \arc\cos(-0) = 3\pi/2$$

where the $\pi/2$ possibility must be rejected, it being clear that r can approach infinity along the trajectory only in the fourth ('lower right') quadrant.

From (5),

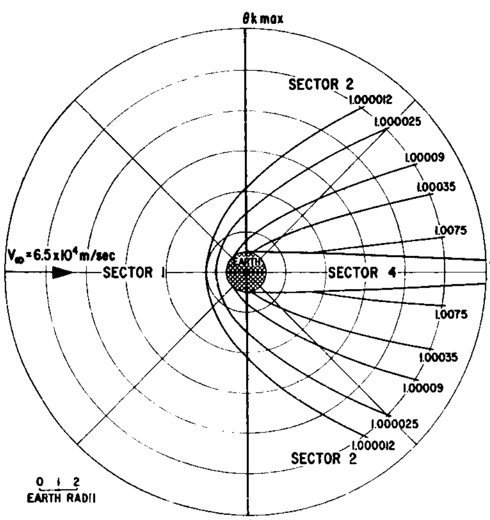

Fig. 9. Total relative particle density contours about a finite attractive center arriving from a unit monodirectional, monoenergetic flux at infinity for $V_\infty = 6.5 \times 10^4$ m/sec. Particle density at infinity assumed to be (1/65) $\times 10^{-3}$ m^{-3}.

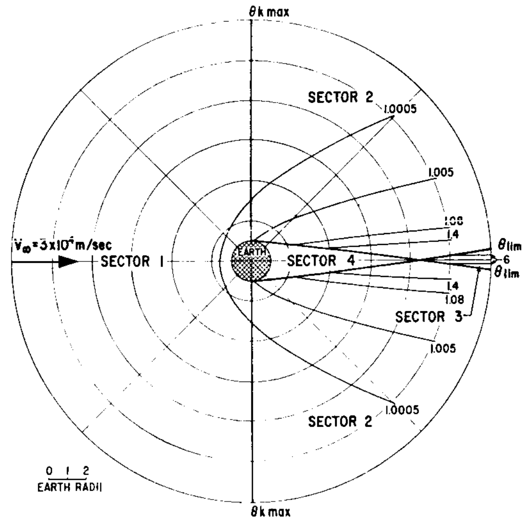

Fig. 10. Total particle density contours, about a finite attractive center, relative to a particle density at infinity of 3.3×10^{-5} m^{-3} arriving from a unit monodirectional, monoenergetic flux at infinity for $V_\infty = 3 \times 10^4$ m/sec.

$$\lim_{r\to\infty} \theta_{\lim} = 2\theta_{k\max} \text{ (on the surface of the center)}$$

$$= 2 \arc\cos \frac{-1}{1 + yr_A} \quad (6)$$

Therefore, we wish to solve for yr_A:

$$\frac{3\pi}{2} = 2 \arc\cos \frac{-1}{1 + yr_A}$$

or

$$\cos\frac{3\pi}{4} = \frac{-1}{\sqrt{2}} = \frac{-1}{1 + yr_A} \quad (7)$$

$$yr_A = \sqrt{2} - 1 = \frac{V_\infty^2}{\gamma M} r_A \quad (8)$$

a general condition relating the speed at infinity, the gravitational constant, and the mass and size of the finite center, for the situation where the $\theta_{k\max}$ and $\theta_{\lim}$ surfaces possess a second intersection at infinity.

If $yr_A < 0.414$ the second intersection occurs at some finite r and θ, specified by (1) and (5) once either y or r_A is given; that is, its location depends on y and r_A, as well as on their product.

For the earth, $r_A \simeq 6.53 \times 10^6$ meters, and therefore the value of y for which a second intersection occurs at infinity is

$$y = 0.414/r_A = 6.34 \times 10^{-8}\,\text{m}^{-1} \qquad (9)$$
$$V_\infty = 5.04 \quad \text{km/sec}$$

Dimensions and Volume of the Null Cone

Any MED possessing nonvanishing kinetic energy at infinity always generates a $\theta_{\lim}$ surface and a node, hence, always generates a null cone. (The case of $V_\infty = 0$ is not included in this discussion; there would be neither a locus of perigee surface or a surface of limiting trajectories, but there would be a null cone.)

The null cone is a volume enclosed by parts of the surface of a skewed hyperboloid of revolution; that is, the axis of rotation for the hyperbola is not the symmetry axis (specified by $\theta = \theta_k$), but is the line through the center and parallel to the velocity vector of the MED at infinity. Its base is that part of the surface of the finite attractive center intercepted by the $\theta_{\lim}$ surface.

Straightforward but tedious integration shows the volume of the null cone to be

$$V_c = \frac{4\pi r_A^3}{3}\begin{cases}\frac{3}{8}(r_Ay + 2) - \frac{r_Ay}{4}\frac{(r_Ay + 4)}{(r_Ay + 1)} + \\ \frac{3}{4}\frac{(r_Ay + 2)}{(r_Ay + 1)}\frac{1}{r_Ay} \\ -\frac{3}{4}\frac{(r_Ay + 2)}{r_A^2y^2}\ln(1 + r_Ay)\end{cases} \qquad (10)$$

where the first factor is the volume of the attractive center, and y of the second factor is $y \equiv V_\infty^2/\gamma M$.

The length l of the null cone, or the distance from the center to the node, can be shown to be

$$l = r_A + (yr_A^2/2) \qquad (11)$$

or in terms of r_A

$$l/r_A = 1 + (yr_A/2) \qquad (12)$$

The height of the null cone above the 'surface' (or atmosphere) is clearly

$$h = l - r_A = \frac{r_A^2y}{2} = \frac{r_A^2V_\infty^2}{2\gamma M} \qquad (13)$$

or, in terms of r_A,

$$h/r_A = r_AV_\infty^2/2\gamma M \qquad (14)$$

Obviously, the extent of the null or shielded cone above the surface of the center varies directly as the kinetic energy of the particles of the MED. Figures 7 and 8 show the relative volumes and heights of null cones formed about an earth-size attractive center for the established spectrum of meteoric speeds.

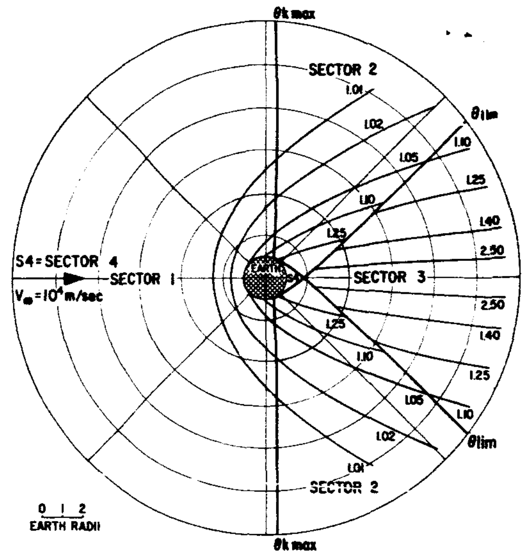

Fig. 11. Total particle density contours, about a finite attractive center, relative to a particle density at infinity of 10^{-4} m^{-3} arriving from a unit monodirectional, monoenergetic flux at infinity for $V_\infty = 10^4$ m/sec.

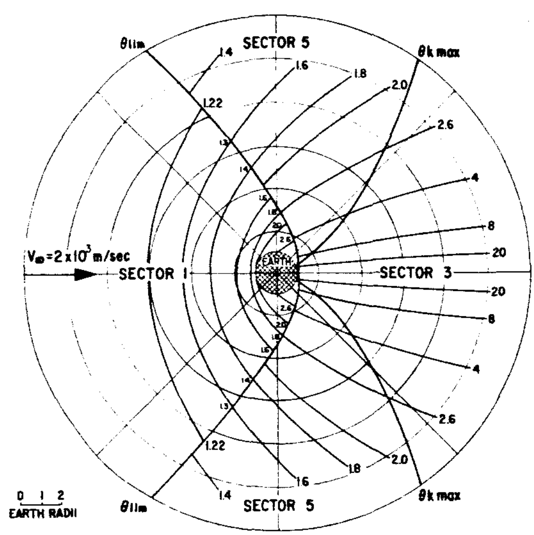

Fig. 12. Total particle density contours, about a finite attractive center, relative to a particle density at infinity of 5×10^{-4} m^{-3} arriving from a unit monodirectional, monoenergetic flux at infinity for $V_\infty = 2 \times 10^3$ m/sec.

Results and Conclusions

Figures 1 through 6 display the detailed flux field patterns resulting when a monodirectional, monoenergetic meteoric stream is incident upon a finite attractive center. Of particular physical interest are the abrupt discontinuities which occur along the θ_{lim} surface; these are prominent in Figures 4 and 5 but not so obvious in Figures 1, 2, and 3 owing to the θ_{lim} surface being of breadth less than the earth out to distances shown in these plots. In Figures 4 and 5 it can also be observed that, the closer one is to the attractive center, the greater the discontinuities observed upon crossing the θ_{lim} surface, say upon a circular path (satellite orbit) concentric with the center. Such discontinuities are observed [*Singer*, 1963; *Rushol*, 1963; *Alexander et al.*, 1961; *Dubin et al.*, 1962; *Dubin and McCracken*, 1962], and the foregoing analysis provides an explanation in the case of unbound particles relative to the earth.

An alternative or conjectural explanation of abrupt variations in particulate fluxes is that the streams are essentially sporadic in time. Very likely, 'clouds' of meteoric or cosmic dust do exist. However, sporadicity alone may be quite an insufficient explanation, in that observed concentrations near the earth may be in excess of those expected in remote space. The theory given here predicts concentrations of flux near attractive centers that are orders of magnitude greater than the fluxes at remote distances.

Although flux is more meaningful than density in most physical discussions, and certainty is the more basic quantity in considerations of meteoric shielding, we have also presented plots of particle density contours for the same V_∞ values used to obtain the flux contour plots of Figures 2 through 5. The associated particle density contours, normalized to unit *flux* at infinity, are exhibited in Figures 9 through 12.

Density is not nearly as significant as flux in phenomena involving high speed particles; indeed, if meteoric hazards due to hypervelocity particles were evaluated on the basis of density data alone, the results could be quite misleading. This deception arises from the fact that, first, the more interesting physical effects (i.e., impact phenomena) are strongly velocity, or flux, dependent, and, second, the density is obtained by dividing the flux by the particle's speed; thus, a significant flux can appear to be an insignificant density.

Some authors, in considering the question of gravitational focusing by attractive centers, have examined only the density, and, finding but a small averaged enhancement, have concluded this effect is of little importance. (Even the density is greatly enhanced, however, if the pattern is examined in detail along the downstream axis, i.e. on the axis behind the center and beyond the null cone.) Fundamentally the confusion arises because the attractive center, in addition to deflecting the trajectories of the incident particles, also causes their speeds to increase as the center is approached; thus, the increase in flux near an attractive center is of necessity always greater than the corresponding increase in density.

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